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# INTERACTING CONTINUOUS MEDIUM COMPOSED OF AN ELASTIC SOLID AND AN INCOMPRESSIBLE NEWTONIAN FLUID†

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**Abstract**—The constitutive equations for an interacting continuum composed of an elastic solid and an incompressible Newtonian fluid are developed. The condition of incompressibility alters the equations previously obtained by Green and Steel. The concept of thermodynamic pressure is introduced and the new equations are compared with those developed by previous investigators.

Using these constitutive equations, methods of solution are presented in terms of displacements or a stress function for the steady state condition. The equations developed are shown to reduce to Darcy's law for flow of fluids through a rigid porous medium and Biot's equation of fluid flow through a linear elastic solid. One and two-dimensional steady-state problems are solved as examples.

## 1. INTRODUCTION

TRUESDELL [1, 2] developed the general framework for heterogeneous media, and gave a comprehensive analysis of four different approaches to a linear theory of diffusion. Using this approach, Adkins [3–6] has formulated a non-linear theory and discussed the invariance requirements and the restrictions imposed upon the constitutive equations. Kelly [7] and Hayday [8] have generalized these theories for chemically reacting media and presented alternate formalisms. In these works, the equations of mass, momentum and energy balance were postulated for each component of the mixture neglecting possible thermodynamic restrictions which might be imposed upon these equations. Green and Naghdi [9] presented a new approach to the problem in which they proposed a single energy equation and an entropy production inequality for the whole continuum allowing for chemical and thermal reactions. By systematic application of invariance requirements, they derived the basic equations.

Green and Steel [10] applied this theory to derive the constitutive equations for a mixture of a Newtonian fluid and an elastic solid and also the mixture of two elastic solids. Following the developments of Green *et al.* this paper will extend their theory to include an incompressibility condition on the fluid. The use of this incompressibility condition alters Green and Steel's equations and allows a comparison between these continuum

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theories and the classical work of Darcy for flow through a porous medium [11–13] and the field equations developed by Biot for flow through deformable solids [14–26].

## 2. BASIC EQUATIONS

The motion of a mixture of two components, $S_1$ and $S_2$, is referred to fixed Cartesian coordinates with material coordinates designated by $X$ and $Y$ respectively. The position of particles at time $t$ is given by

$$x_i = x_i(X_j, t) \qquad y_i = y_i(Y_j, t). \tag{2.1}$$

These particles are considered to occupy the same position at time $t$ so that

$$y_i = x_i. \tag{2.2}$$

The velocity and acceleration vectors of $S_1$ are given by $u_i$ and $a_i$ and those of $S_2$ by $v_i$ and $g_i$. The densities at time $t$ are $\rho_1$ and $\rho_2$ and the rate of deformation tensors are defined to be

$$d_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}) \qquad f_{ij} = \tfrac{1}{2}(v_{i,j}+v_{j,i}) \tag{2.3}$$

where a comma denotes partial differentiation with respect to $x_i$ or $y_i$.

The vorticity tensors and other mixture variables are defined as follows:

$$\begin{array}{lll} \Gamma_{ij} = \tfrac{1}{2}(u_{i,j}-u_{j,i}) & \Lambda_{ij} = \tfrac{1}{2}(v_{i,j}-v_{j,i}) & \rho = \rho_1+\rho_2 \\ \rho v_i = \rho_1 u_i + \rho_2 v_i & \dfrac{D}{Dt} = \dfrac{\partial}{\partial t}+\bar{v}_m\dfrac{\partial}{\partial x_m} & \rho\dfrac{D}{Dt} = \rho_1\dfrac{^{(1)}D}{Dt}+\rho\dfrac{^{(2)}D}{Dt} \end{array} \tag{2.4}$$

where the numerical superscript on the material time derivative refers to the component in question. The summation convention is being used in the above equations and throughout the paper.

The basic balance equations have been developed previously and are presented here for reference only. The continuity equations for a binary mixture in the absence of chemical reactions are as follows

$$\frac{^{(1)}D\rho_1}{Dt}+\rho_1 u_{k,k} = 0 \qquad \frac{^{(2)}D\rho_2}{Dt}+\rho_2 v_{k,k} = 0. \tag{2.5}$$

The equations of motion for the mixture are

$$(\sigma_{ki}+\tau_{ki})_{,k}+\rho_1 F_i+\rho_2 G_i = \rho_1 a_i+\rho_2 g_i \tag{2.6}$$

where $\sigma_{ki}$ and $\tau_{ki}$ are partial stresses of the solid and the fluid respectively and $F_i$ and $G_i$ are the body forces per unit mass of each continuum. The partial stresses satisfy the following symmetry relation

$$\sigma_{ki}+\tau_{ki} = \sigma_{ik}+\tau_{ik}. \tag{2.7}$$

The diffusive force $\pi_i$ is given by

$$\pi_i = \tfrac{1}{2}(\sigma_{ki}-\tau_{ki})_{,k}+\tfrac{1}{2}\rho_1(F_i-a_i)-\tfrac{1}{2}\rho_2(G_i-g_i). \tag{2.8}$$

The entropy production inequality, under isothermal conditions, may be written [10]

$$-\rho\frac{DA}{Dt}+\pi_i(u_i-v_i)+\tfrac{1}{2}(\sigma_{ki}+\sigma_{ik})d_{ik}+\tfrac{1}{2}(\tau_{ki}+\tau_{ik})f_{ik}+\tfrac{1}{2}(\sigma_{ki}-\sigma_{ik})(\Gamma_{ik}-\Lambda_{ik})\geq 0 \tag{2.9}$$

where $A = U-TS$ is the Helmholtz free energy, $U$ is the internal energy per unit mass, $S$ is the entropy per unit mass and $T$ is the temperature.

## 3. INCOMPRESSIBILITY CONDITION

The following assumptions will be used in this section and throughout the paper.

(a) Both the elastic solid and the fluid are initially at rest under zero initial stresses.

(b) The continuum is initially homogeneous and isotropic and undergoes an isothermal deformation.

(c) The displacement of the solid, the change in density and velocity as well as their space and time derivatives remain small during the motion so that higher order terms may be neglected. Although the field equations will be developed and methods of solution proposed for a continuum composed of an incompressible Newtonian fluid and a solid, the incompressibility condition will be presented for the more general case of $n$ incompressible fluids and a linear elastic solid. The initial porosity of the solid is designated by $P_0$ and the initial volume concentrations of the fluids within the pores of the solid by $C^\alpha$, $\alpha = 1, 2, 3, \ldots n$. It may be noted that

$$\sum_{\alpha=1}^{n} C^\alpha = 1. \tag{3.1}$$

Let $\bar{V}_1$ be the initial volume of an element of the solid and $V_1$ the volume of the same element at $t = t$. The following relation may be seen to hold

$$V_1 = \bar{V}_1(1+e_{mm}) \tag{3.2}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial\omega_i}{\partial x_j}+\frac{\partial\omega_j}{\partial x_i}\right) \tag{3.3}$$

$$\omega_i = x_i - X_i. \tag{3.4}$$

There is a linear relationship between the compressibility and the volume change of the pores which may be written as

$$V = R(V_1-\bar{V}_1)+P_0\bar{V}_1 \tag{3.5}$$

or

$$V = \bar{V}_1(Re_{mm}+P_0) \tag{3.6}$$

where $V$ is the actual volume of the pores in an element $\bar{V}_1$ at time $t$ (only interconnected pores are considered and the closed pores are treated as part of the solid). $R$ is a constant expressing the ratio of pore compressibility to the total compressibility.

If $\rho_0^\alpha$, $\rho_i^\alpha$, $\rho_t^\alpha$ and $V^\alpha$ are the initial density of the fluid $\alpha$, the initial density of the fluid $\alpha$ in the mixture, the density of the fluid $\alpha$ in the mixture at time $t$ and the actual volume of the fluid $\alpha$ within the volume $V_1$ respectively, then

$$\rho_0^\alpha V^\alpha = \rho_t^\alpha V_1 \qquad \text{for } \alpha = 1, 2, \ldots n \tag{3.7}$$

and

$$\sum_{\alpha=1}^{n} V^\alpha = V. \tag{3.8}$$

Also, it may be noted that

$$\rho_i^\alpha = C^\alpha P_0 \rho_0^\alpha \qquad \text{for } \alpha = 1, 2, \ldots n. \tag{3.9}$$

By use of equations (3.2), (3.5), (3.7), (3.9) in (3.8), one obtains

$$P_0 \sum_{\alpha=1}^{n} \frac{C^\alpha \rho_t^\alpha}{\rho_i^\alpha} = (P - P_0) e_{mm} + P_0. \tag{3.10}$$

The continuity equations for the fluids can be written as

$$\frac{\partial \eta^\alpha}{\partial t} + \rho_i^\alpha f_{kk}^\alpha = 0 \qquad \text{for } \alpha = 1, 2, \ldots n \tag{3.11}$$

where $f_{ij}^\alpha$ is the rate of deformation tensor for the $\alpha$th fluid and

$$\eta^\alpha = \rho_t^\alpha - \rho_i^\alpha \qquad \text{for } \alpha = 1, 2, \ldots n \tag{3.12}$$

$$\frac{\partial \eta^\alpha}{\partial t} = \frac{\partial \rho_t^\alpha}{\partial t} \qquad \text{for } \alpha = 1, 2, \ldots n. \tag{3.13}$$

If equation (3.13) and (3.10) are substituted into the partial time derivative of equation (3.10), one obtains

$$P_0 \sum_{\alpha=1}^{n} C^\alpha f_{kk}^\alpha + (P - P_0) \frac{\partial e_{mm}}{\partial t} = 0. \tag{3.14}$$

In the case of a binary mixture of a fluid and an elastic solid $\alpha = 1$ and $C^1 = 1$, therefore equation (3.14) reduces to

$$f_{kk} = \frac{P_0 - R}{P_0} \frac{\partial e_{mm}}{\partial t} \tag{3.15}$$

$$0 < R \leq 1 \qquad 0 < P_0 \leq 1$$

where $f_{ij}$ is the rate of deformation tensor of the fluid. Biot [17] has given an incompressibility condition for a mixture of a solid modeled by rigid spheres connected by helical springs and an incompressible fluid. This relation can be obtained from equation (3.15) by setting $R = 1$ and integrating the equation.

The incompressibility condition (3.15) reduces to

$$f_{kk} = 0 \tag{3.16}$$

in the following cases:

(a) steady-state case where $\partial e_{mm}/\partial t = 0$

(b) the solid is rigid so $e_{mm} = 0$.

## 4. CONSTITUTIVE EQUATIONS

The constitutive equations for a mixture of an incompressible Newtonian fluid and an elastic solid may be obtained by applying the incompressibility constraint to the equations obtained by Green and Steel [10] for a Newtonian fluid and a linear elastic solid. In this case, the entropy production inequality, as given in [10], should hold provided it satisfies the constraint condition (3.15).

If the quantity

$$\bar{p}\left[f_{kk}+\frac{R-P_0}{P_0}d_{kk}\right] = \bar{p}[f_{ik}+a_1 d_{ik}]\delta_{ik}, \tag{4.1}$$

where $a_1 = (R-P_0)/P_0$ and $\bar{p}$ is an arbitrary scalar function (Lagrange multiplier), is added to entropy production inequality, the resulting equation may be treated as a variational problem. This approach, similar to that used by Mills [27], yields:

$$\sigma_{ik} = \sigma_{ki} = \frac{1}{2}\rho\frac{\partial x_i}{\partial X_r}\frac{\partial x_k}{\partial X_s}\left(\frac{\partial A}{\partial e_{rs}}+\frac{\partial A}{\partial e_{sr}}\right)+a_1\bar{p}\delta_{ik} \tag{4.2}$$

$$\tau_{ki} = \tau_{ik} = -\rho\rho_2\frac{\partial A}{\partial \rho_2}\delta_{ik}+\bar{p}\delta_{ik}+\lambda f_{rr}\delta_{ik}+2\mu f_{ik} \tag{4.3}$$

$$\pi_i = \rho_1\frac{\partial A}{\partial \rho_2}\frac{\partial \rho_2}{\partial y_i}-\frac{1}{2}\rho_2\left(\frac{\partial A}{\partial e_{rs}}+\frac{\partial A}{\partial e_{sr}}\right)\frac{\partial e_{rs}}{\partial x_i}+a(u_i-v_i) \tag{4.4}$$

where

$$\mu \geq 0 \qquad \lambda+\tfrac{2}{3}\mu \geq 0 \qquad a \geq 0. \tag{4.5}$$

The Taylor series expansion for the Helmholtz free energy may be written

$$\bar{\rho}A = A_0+\alpha_1 e_{mm}+\alpha_{2\eta}+\tfrac{1}{2}\alpha_4 e_{mm}e_{nn}+\alpha_5 e_{mn}e_{mn}+\tfrac{1}{2}\alpha_6\eta^2+\alpha_8 e_{mm}\eta \tag{4.6}$$

where $\bar{\rho} = \bar{\rho}_1+\bar{\rho}_2$ and $\eta = \rho-\rho_0$. The constants $A_0, \alpha_1 \ldots$ depends upon initial densities of each substance.

The assumption of zero initial stress implies that $A_0 = \alpha_1 = \alpha_2 = 0$. The variable $\eta$ in the equation (4.6) can be also related to the dilatation of the solid, $e_{mm}$, by use of (3.11) and (3.15). By incorporating the above remarks into (4.6) and substituting it into constitutive equations yields

$$\sigma_{ki} = \sigma_{ik} = -a_1 p\delta_{ik}+2a_2 e_{ik}+a_3 e_{mm}\delta_{ik} \tag{4.7}$$

$$\tau_{ki} = \tau_{ik} = -p\delta_{ik}+\lambda f_{rr}\delta_{ik}+2\mu f_{ik} \tag{4.8}$$

$$\pi_i = a(U_i-V_i) \tag{4.9}$$

where

$$a_2 = \alpha_5 \qquad a_1 = \frac{P_0-R}{P_0}$$

$$a_3 = \alpha_4+\bar{\rho}_2 a_1\alpha_8+a_1^2\bar{\rho}_2^2\alpha_6+a_1\bar{\rho}_2\alpha_8$$

and

$$-p = \frac{1}{a_1}[a_1\bar{p} - a_1(\bar{\rho}_2\alpha_8 + \bar{\rho}_2^2 a_1\alpha_6)e_{mm}]. \tag{4.10}$$

The scalar function $p$ is equivalent to the thermodynamic pressure defined for a simple media. Similar constitutive equations, in the absence of the incompressibility constraint, were obtained by Green and Steel [10]. If, in their equations, an equivalent thermodynamic pressure is defined as

$$p = \bar{\rho}_2\alpha_6\eta + \bar{\rho}_2\alpha_8 e_{mm}, \tag{4.11}$$

then analogy to the theory of simple Newtonian fluids shows that in the case of a compressible fluid this definition serves as an equation of state. In the case of incompressibility, $p$ introduces a new unknown into the system of equations together with an additional equation; namely, the incompressibility condition.

It may be seen that the constitutive equation of the solid is coupled with that of the fluid through the thermodynamic pressure, while the partial stresses of the fluid are coupled with those of the solid by the solid dilatation through the equation of state or the incompressibility condition. It should be kept in mind that the coefficients in these equations are not numerically the same as those of the corresponding single medium. However, we may notice that all the equations must reduce to those for a single elastic solid or fluid when $\rho_2$ or $\rho_1$ vanishes, respectively.

## 5. DISPLACEMENT EQUATION

If we substitute the rate of deformation and strain tensor in terms of the displacements and velocities into the equations (4.7)–(4.9), and combining with equations of motion and (3.15), we obtain

$$-a_1\nabla p + (a_2 + a_3)\nabla(\nabla \cdot \boldsymbol{\omega}) + a_2\nabla^2\boldsymbol{\omega} - a(\mathbf{U} - \mathbf{V}) = \bar{\rho}_1\frac{\partial^2\ddot{\boldsymbol{\omega}}}{\partial t} \tag{5.1}$$

$$-\nabla p + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{V}) + \mu\nabla^2\mathbf{V} + a(\mathbf{U} - \mathbf{V}) = \bar{\rho}_2\frac{\partial \mathbf{V}}{\partial t} \tag{5.2}$$

$$\nabla \cdot \mathbf{V} = a_1\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{\omega} \tag{5.3}$$

where the body forces (external) are neglected. The above equations are written in tensorial form where $\nabla$ denotes the conventional del operator.

In the steady-state case the time derivatives of dependent variables vanish, yielding

$$-a_1\nabla p + (a_2 + a_3)\nabla(\nabla \cdot \boldsymbol{\omega}) + a_2\nabla^2\boldsymbol{\omega} + a\mathbf{V} = 0 \tag{5.4}$$

$$-\nabla p + \mu\nabla^2\mathbf{V} - a\mathbf{V} = 0 \tag{5.5}$$

$$\nabla \cdot \mathbf{V} = 0 \tag{5.6}$$

where use has been made of the equation (5.6) in (5.5). The equations (5.4)–(5.6) constitute seven differential equations for seven unknowns, $\mathbf{V}$, $\boldsymbol{\omega}$ and $p$. By application of $\nabla$ operator to equation (5.5) and making use of (5.6), one obtains:

$$\nabla^2 p = 0. \tag{5.7}$$

The general solution of $\mathbf{V}$ consists of the solution of the reduced equation

$$\mu\nabla^2\mathbf{V} - a\mathbf{V} = 0 \tag{5.8}$$

plus the particular integral of equation (5.5). If the general solution of (5.8) is denoted by $\mathbf{V}^{(r)}$, and a particular solution for the equation (5.5) may be written

$$\mathbf{V}\text{ particular} = -\frac{1}{a}\nabla p$$

then the complete solution for (5.5) is

$$\mathbf{V} = \mathbf{V}^{(r)} - \frac{1}{a}\nabla p. \tag{5.9}$$

The general solution $\mathbf{V}^{(r)}$ must also satisfy the condition (5.6). In the two dimensional case, equation (5.6) reduces to

$$\frac{\partial V_x^{(r)}}{\partial x} + \frac{\partial V_y^{(r)}}{\partial y} = 0. \tag{5.10}$$

It then follows that the velocity vector $V^{(r)}$ can be derived from a scalar function $\psi$ such that:

$$V_x^{(r)} = \frac{\partial\psi}{\partial y} \qquad V_y^{(r)} = -\frac{\partial\psi}{\partial x}. \tag{5.11}$$

Substituting for $V_x^{(r)}$ and $V_y^{(r)}$ into the equation (5.8) yields:

$$\frac{\partial}{\partial y}[\mu\nabla^2\psi - a\psi] = 0 \tag{5.12}$$

$$\frac{\partial}{\partial x}[\mu\nabla^2\psi - a\psi] = 0. \tag{5.13}$$

The above pair of equations imply that the expression inside the bracket is a constant. This constant can be assumed to be zero without any loss of generality in the velocity solution. Therefore, the problem reduces to finding a function $\psi$ satisfying the Helmholtz equation,

$$\mu\nabla^2\psi - a\psi = 0. \tag{5.14}$$

It may be seen that the general solution of the velocity field consists of the linear combination of two scalar functions which satisfy the Laplace and Helmholtz equations. In order to obtain a general solution for the displacements of the solid, we add equation (5.4) and (5.5) yielding

$$-(a_1+1)\nabla p + (a_2+a_3)\nabla(\nabla\cdot\boldsymbol{\omega}) + a_2\nabla^2\boldsymbol{\omega} + \mu\nabla^2\mathbf{V} = 0. \tag{5.15}$$

Equation (5.15) shows that the vector function $a_2 V^2\boldsymbol{\omega} + \mu\nabla^2\mathbf{V}$ is irrotational (referring to Helmholtz's representation) and hence can be expressed as the gradient of a scalar function $\phi_1$,

$$a_2\nabla^2\boldsymbol{\omega}+\mu\nabla^2\mathbf{V} = \nabla\phi_1 \tag{5.16}$$

where without loss of generality and under sufficient smoothness and integrability conditions we can find another scalar function $\phi$ such that

$$\phi_1 = \nabla^2\phi. \tag{5.17}$$

Noting

$$\nabla\nabla^{2(\,)} = \nabla^2\nabla(\,) \tag{5.18}$$

equation (5.16) may be written as

$$\nabla^2[a_2\boldsymbol{\omega}+\mu\mathbf{V}-\nabla\phi] = 0. \tag{5.19}$$

Let

$$a_2\boldsymbol{\omega}+\mu\mathbf{V}-\nabla\phi = \boldsymbol{\Gamma}, \tag{5.20}$$

where $\boldsymbol{\Gamma}$ is a vector function which satisfied

$$\nabla^2\boldsymbol{\Gamma} = 0. \tag{5.21}$$

The general solution of $\omega$ is

$$\boldsymbol{\omega} = \frac{1}{a_2}(\boldsymbol{\Gamma}+\nabla\phi)-\frac{\mu}{a_2}\mathbf{V}. \tag{5.22}$$

Taking the divergence of equation (5.4) and making use of (5.6) and (5.7), one obtains

$$\nabla^2(\nabla\cdot\boldsymbol{\omega}) = 0. \tag{5.23}$$

Combining (5.15)–(5.17) gives:

$$-(a_1+1)\nabla p+(a_2+a_3)\nabla(\nabla\cdot\boldsymbol{\omega})+\nabla\nabla^2\phi = 0. \tag{5.24}$$

Again taking the divergence of equation (5.24) and using the relations (5.7) and (5.23) yields

$$\nabla^4\phi = 0. \tag{5.25}$$

Hence $\phi$ is a biharmonic scalar function, and the general solution of the system of equations (5.4)–(5.6) reduces to a linear combination of the general solutions of some classical equations whose properties are well established. The solution to a particular problem would be obtained by choice of these functions such that they satisfy the prescribed boundary conditions. Note that $\phi$ and $\boldsymbol{\Gamma}$ are not independent and must satisfy the condition:

$$\nabla^2\phi = \frac{(a_1+1)a_2}{2a_2+a_3}p-\frac{a_2+a_3}{2a_2+a_3}\nabla\cdot\boldsymbol{\Gamma}. \tag{5.26}$$

The above relation obtained by substituting the $\boldsymbol{\omega}$ from (5.22) into (5.15) and making use of (5.6).

## 6. SOLUTION OF THE STEADY-STATE PLANE STRAIN PROBLEMS

In this section, the displacements of the solid as well as the velocities of the fluid are in $x$–$y$ plane and independent of $z$ coordinate. It was shown, in the previous section, that the equation governing the motion of the fluid is independent of the solid deformations and therefore can be solved independently. The equation governing the solid deformations, however, contain fluid velocity terms. Once the velocity of the fluid is obtained from the governing Helmholtz equation, the stresses in the solid may be obtained by use of a stress function. The solid equilibrium equations, in the absence of external body forces are,

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+av_x = 0 \tag{6.1}$$

$$\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{xy}}{\partial x}+av_y = 0. \tag{6.2}$$

The above equations are identically satisfied if

$$\sigma_{xx} = p+\frac{\partial^2 \theta}{\partial y^2}-a\int^x \frac{\partial \psi}{\partial y}\,\mathrm{d}x \tag{6.3}$$

$$\sigma_{yy} = p+\frac{\partial^2 \theta}{\partial x^2}+a\int^y \frac{\partial \psi}{\partial x}\,\mathrm{d}y \tag{6.4}$$

$$\sigma_{xy} = -\frac{\partial^2 \theta}{\partial x \partial y} \tag{6.5}$$

where $\theta$ is the stress function for solid and $\psi$ is as defined by (5.11). Again, as in the theory of elasticity, the stress function $\theta$ is the only unknown function, but the compatibility relations place a condition on the otherwise arbitrary stress function. The compatibility conditions of the present theory are identical to those of linear elasticity, and for plane strain, the only remaining equation is:

$$\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y} \tag{6.6}$$

Substituting for $e_{xx}$, $e_{yy}$ and $e_{xy}$ from constitutive relations in term of the stresses and substituting for stresses from (6.3)–(6.5) and making use of (5.7) yields

$$\frac{1}{a}\nabla^4 \theta = \int^x \frac{\partial^3 \psi}{\partial y^3}\,\mathrm{d}x - \int^y \frac{\partial^3 \psi}{\partial x^3}\,\mathrm{d}y. \tag{6.7}$$

The general solution for $\theta$ is obtained by addition of the particular solution of (6.7) to a biharmonic scalar function. The arbitrary constants will be chosen to satisfy the prescribed boundary conditions.

## 7. DISCUSSION AND EVALUATION OF THEORIES

It is beneficial to examine Green and Steel's theory and the work presented here in comparison with that of Darcy's for fluid flow through underformable porous media and the Biot theory [14–25] for flow of fluids through deformable media.

Darcy's law asserts that, macroscopically the velocity is proportional to the pressure gradient acting on the fluid

$$\mathbf{V} = -\frac{k}{\mu}\nabla p \tag{7.1}$$

where $\mu$ is the viscosity of the fluid and $k$ is the permeability of the solid. The permeability, $k$, in the above equation has dimension of length squared and expresses the ease of the fluid flow through porous media. The monographs and the literature on the field give detailed discussions of the permeability and the various formulas expressing it in terms of porosity and other variables.

This law together with the equation of state and the continuity equation constitute a complete system of equations. These equations supplemented by initial and boundary conditions provide all the necessary information for the solution of any particular problem. For incompressible fluids, the equations of state and continuity reduce to

$$\rho = \text{const.} \tag{7.2}$$

and hence

$$\nabla^2 p = 0. \tag{7.3}$$

According to the above equations, there is no distinction between steady-state and non-steady-state problems for incompressible fluids. In order to compare the above fundamental equations to those of the present theory, the solid will be assumed to be undeformable and hence $\boldsymbol{\omega} = \mathbf{u} \equiv 0$. The equations (5.2) and (5.3) reduce to

$$-\frac{\mu}{a}\nabla^2\mathbf{V}+\mathbf{V} = -\frac{1}{a}\nabla p \tag{7.4}$$

$$\nabla\cdot\mathbf{V} = 0 \qquad \nabla^2 p = 0. \tag{7.5}$$

In order to reduce (7.4) and (7.1), the diffusive coefficient $a$ should be assumed as

$$a = \frac{\mu}{k}. \tag{7.6}$$

This implies that the diffusive force vanished for ideal fluids. Substituting (7.6) and (7.4) yields

$$-k\nabla^2\mathbf{V}+\mathbf{V} = \frac{k}{\mu}\nabla p. \tag{7.7}$$

The above equation can be reduced to (7.1) if the term $k\nabla^2\mathbf{V}$ is negligible compared to $\mathbf{V}$. Since the velocity as well as its space derivatives are of the same order, $k$ has to be small to perform this reduction. For many materials this is the case.

However, on the other extreme, if $k$ is large, equation (7.7) becomes

$$-\mu\nabla^2\mathbf{V} = \nabla p \tag{7.8}$$

which is the equation for slow viscous flow of fluids. The reduced equation (7.4) with the special choice of $a$ from (7.6) includes two different extremes, namely, a pure slow viscous flow of fluids, and flow of fluid through highly impermeable materials.

Atkin *et al.* [28] has given one possible set of boundary conditions for which the problem has a unique solution. He noted that at each point on the boundary two vector boundary conditions and a scalar function for thermal considerations must be specified. It is easily seen from equations (7.4) and (7.5) and the remarks in [28], that in the steady-state case, a vector boundary condition should be prescribed at each point on the boundary for the fluid part only. This may be noted also from purely physical considerations.

Contrary to the above remarks the Darcy equation does not allow specification of a vector boundary condition but only a scalar function. It may be seen that this occurred because of neglect of the term $k\nabla^2\mathbf{V}$. In the case of low permeability, the error arising from the above simplification is insignificant far from the boundary while the error might be quite serious near and at the boundaries.

Crochet and Neghdi [29] derived a one-dimensional equation for uniform flow through porous media which is similar to Darcy's law. They point out that a general form of Darcy's law can be obtained from the constitutive equations and the use of other field equations. The equation (7.7) was obtained in a similar manner and can be considered as a generalized Darcy's law. It is obvious that by relaxing some of the assumptions used here, a more general law could be obtained.

## 8. BIOT'S THEORY

A major extension of the classical theory of flow through an elastically deformable media was presented by Biot [14–25]. He obtained the following constitutive equations for the stresses

$$\sigma_{ij} = 2Ne_{ij}+Me_{kk}\delta_{ij}+Q\varepsilon\delta_{ij} \tag{8.1}$$

$$\tau_{ij} = (Qe_{kk}+L\varepsilon)\delta_{ij} = \sigma\delta_{ij} \tag{8.2}$$

where $\varepsilon$ is the dilation of the fluid by

$$\varepsilon = \nabla\cdot\boldsymbol{\omega}_p \tag{8.3}$$

and $\boldsymbol{\omega}_p$ is the fluid displacement vector. $\sigma$ equals $-\beta p$, where $\beta$ is the fraction of fluid element per unit section and $p$ is the fluid pressure. The equation of motion in the absence of body forces for the quasi-static theory is

$$(\sigma_{ij}+\sigma\delta_{ij})_{,j} = 0 \tag{8.4}$$

and the modified Darcy's law is

$$\nabla\sigma = a(\mathbf{V}-\mathbf{U}). \tag{8.5}$$

For the dynamic theory [20], the equations become:

$$\sigma_{ij,j} = \frac{\partial}{\partial t}(\rho_{11}u_i+\rho_{12}v_i)-a(v_i-u_i) \tag{8.6}$$

$$\sigma_{,i} = \frac{\partial}{\partial t}(\rho_{12}u_i+\rho_{22}v_i)+a(v_i-u_i) \tag{8.7}$$

where $\rho_{11}+\rho_{12} = \rho_1, \rho_{22}+\rho_{12} = \rho_2, \rho_{12}$ is a mass coupling parameter. The interacting continuum theory can be reduced to Biot's theory if terms with viscosity coefficients $\lambda$ and $\mu$ are eliminated from the constitutive equation for the fluid. This can be justified if the viscosity is low enough such that the only dominating term in the expression for the fluid stress would be the hydrostatic pressure.

In the dynamic theory, equations (8.6), (8.7) can be written in the following form:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho_1 \frac{\partial u_i}{\partial t} + \rho_{12} \frac{\partial}{\partial t}(v_i - u_i) - a(v_i - u_i) \tag{8.8}$$

$$\frac{\partial \sigma}{\partial x_i} = \rho_2 \frac{\partial v_i}{\partial t} - \rho_{12} \frac{\partial}{\partial t}(v_i - u_i) + a(v_i - u_i). \tag{8.9}$$

From the standpoint of Green and Naghdi's theory, it may be noted that the diffusive force according to Biot's formula is

$$\pi_i = \rho_{12} \frac{\partial}{\partial t}(v_i - u_i) - a(v_i - u_i). \tag{8.10}$$

Thus in Biot's dynamic theory, the diffusive force is the same as that of the quasi-static plus the term $\rho_{12}(\partial/\partial t)(v_i - u_i)$. This last term does not satisfy the requirement of frame indifference. Its existence, however, can be justified if the diffusive force in the corresponding non-linear theory has this term as one of its linear terms.

## 9. ONE-DIMENSIONAL PROBLEM OF INFINITE PLATE

As an example of this theory, consider the infinite plate resting on a rigid highly permeable medium and bounded at $x = 0$ and $x = h$. A constant fluid pressure $p_0$ is applied to the face $x = 0$. Assume that the lateral displacements can be neglected and the only non-vanishing components of the velocity and displacement vectors are

$$\omega_x = \omega_x(x) \qquad v_x = v_x(x) \tag{9.1}$$

and the boundary conditions are:

$$\text{at } x = 0 \qquad \sigma_{xx} + \tau_{xx} = -p_0 \tag{9.2}$$

$$\text{at } x = h \qquad \omega_x = 0 \qquad p = 0. \tag{9.3}$$

Equation (5.10) reduces to:

$$\frac{\partial^2 p}{\partial x^2} = 0. \tag{9.4}$$

Integrating the above equation and using the boundary condition $(9.3)_2$ yields

$$P = C_0(x - h) \tag{9.5}$$

where $C_0$ is an arbitrary constant. The relations (5.6) and (5.5) give

$$v_x = -\frac{C_0}{a}. \tag{9.6}$$

Using the results (9.5), (9.6) and (5.4) and integrating, gives

$$\omega_x = \frac{(a_1 + 1)C_0}{2(2a_2 + a_3)}(x^2 - h^2) + C_1(x - h) \tag{9.7}$$

where use has been made of the boundary condition $(9.3)_1$, and $C_1$ is a new arbitrary constant.

The fluid stresses are as follows:

$$\tau_{ij} = -C_0(x-h)\delta_{ij}. \tag{9.8}$$

The solid stresses are

$$\sigma_{xx} = (a_3+2a_2)\frac{\partial \omega_x}{\partial x} - a_1 p \tag{9.9}$$

$$\sigma_{yy} = \sigma_{zz} = -a_1 p + a_3 \frac{\partial \omega_x}{\partial x} \tag{9.10}$$

$$\sigma_{ij} = 0 \qquad \text{if } i \neq j. \tag{9.11}$$

Substituting for $\omega_1$ and $p$ yields

$$\sigma_{xx} = C_0(x+a_1 h) + C_1(2a_2+a_3) \tag{9.12}$$

$$\sigma_{yy} = \sigma_{zz} = -a_1 C_0(x-h) + \frac{(a_1+1)a_3 C_0}{(2a_2+a_3)} + C_1 a_3. \tag{9.13}$$

The only remaining boundary condition (9.2) gives the constant $C_1$ in terms of $C_0$:

$$C_1 = \frac{-P_0 + C_0 h(1+a_1)}{(2a_2+a_3)}. \tag{9.14}$$

It may be seen that the solution is indeterminate within a constant $C_0$. However, this indeterminacy may be removed by specifying the surface porosity at the face $x = 0$, and therefore prescribing the separate values of $\sigma_{xx}$ and $p$ at that face.

## 10. SEMI-INFINITE STRIP PROBLEM

As another illustration, consider an infinitely long strip of an elastic solid with width $\pi$. The Cartesian system $(x, y, z)$ will be taken as shown in Fig. 1.

A fluid pressure $p$ is applied to the surface $y = 0$. Under the above conditions it is conceivable to assume all variables to be functions of $x$ and $y$ only.

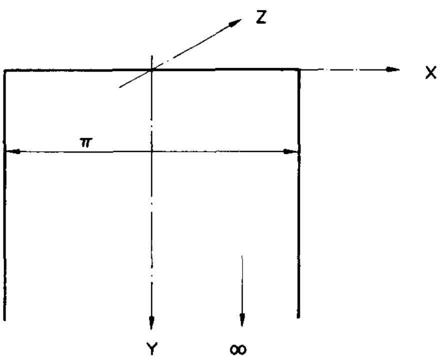


FIG. 1. Semi-infinite strip.

The boundary conditions are

$$\text{at } y = 0: p = A_1 \cos x, v_x = 0, \sigma_{yy} = -A_2 \cos x, \sigma_{xy} = 0 \tag{10.1}$$

$$\text{at } x = \pm\frac{\pi}{2}: p = 0, v_y = f(y), \sigma_{xx} = 0, \sigma_{xy} = 0 \tag{10.2}$$

$$\text{at } y = \infty : p = 0, v_x = v_y = 0. \tag{10.3}$$

The solution for a general fluid pressure at $y = 0$ may be obtained in a similar manner by use of Fourier analysis. The constants $A_1$ and $A_2$ depend on fluid pressure and porosity factor $P_0$. Equation (5.7) reduces to

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0. \tag{10.4}$$

Since $p$ is even in $x$ and decaying in $y$, the solution of equation (10.4) has the following form:

$$p = A e^{-\beta y} \cos \beta x. \tag{10.5}$$

Applying the boundary conditions $(10.1)_1$ and $(10.2)_1$ gives the particular solution for $p$ to be:

$$p = A_1 e^{-y} \cos x. \tag{10.6}$$

Equation (5.18) reduces to

$$\mu\left\{\frac{\partial^2 \psi(x, y)}{\partial x^2} + \frac{\partial^2 \psi(x, y)}{\partial y^2}\right\} - a\psi(x, y) = 0. \tag{10.7}$$

Noting that the velocity components $v_x$ and $v_y$ are decaying in $y$ and are odd and even in $x$ respectively, the following form may be assumed for $\psi$:

$$\psi = B_\alpha e^{-(\alpha^2 + a/\mu)^{1/2} y} \sin \alpha x + \int_0^\infty C_\gamma \cos \gamma y \,\text{sh}\left(\gamma^2 + \frac{a}{\mu}\right)^{\frac{1}{2}} x \, d\gamma. \tag{10.8}$$

The velocity components of the fluid are obtained from (5.9), where use has been made of (10.8) and (10.7) and are as follows:

$$v_x = \frac{1}{a} A_1 e^{-y} \sin x - B_\alpha \left(\alpha^2 + \frac{a}{\mu}\right)^{\frac{1}{2}} e^{-(\alpha^2 + a/\mu)^{1/2}} y \sin \alpha x - \int_0^\infty C_\gamma \gamma \sin \gamma y \,\text{sh} \sqrt{\left(\gamma^2 + \frac{a}{\mu}\right)} x \, d\gamma \tag{10.9}$$

$$v_y = \frac{A_1}{a} e^{-y} \cos x - B_\alpha \alpha e^{-(\alpha^2 + a/\mu)^{1/2} y} \cos \alpha x - \int_0^\infty C_\gamma \left(\gamma^2 + \frac{a}{\mu}\right)^{\frac{1}{2}} \cos \gamma y \,\text{ch} \sqrt{\left(\gamma^2 + \frac{a}{\mu}\right)} x \, d\gamma. \tag{10.10}$$

Application of boundary conditions $(10.1)_2$ yields

$$\alpha = 1 \quad \text{and} \quad B_\alpha \left(1 + \frac{a}{\mu}\right)^{\frac{1}{2}} = \frac{A_1}{a}. \tag{10.11}$$

The only condition on the fluid velocity remaining to be satisfied is $(10.2)_2$. The constant $C_\gamma$ should be chosen such that

$$f(y) = -\int_0^\infty C_\gamma\left(\gamma^2+\frac{a}{\mu}\right)^{\frac{1}{2}} \mathrm{ch}\sqrt{\left(\gamma^2+\frac{a}{\mu}\right)}\frac{\pi}{2}\,\mathrm{d}\gamma. \tag{10.12}$$

Consider the case where $f(y) = 0$ and, therefore, $C_\gamma = 0$. The velocity components become

$$v_x = \frac{A_1}{a}[\mathrm{e}^{-y}-\mathrm{e}^{-(1+a/\mu)^{1/2}y}]\sin x \tag{10.13}$$

$$v_y = \frac{A_1}{a}\left[\mathrm{e}^{-y}-\left(1+\frac{a}{\mu}\right)^{-\frac{1}{2}}\mathrm{e}^{-(1+a/\mu)^{1/2}y}\right]\cos x. \tag{10.14}$$

The fluid stresses become

$$\tau_{xx} = A_1\left[\left(\frac{2\mu}{a}-1\right)\mathrm{e}^{-y}-\frac{2\mu}{a}\mathrm{e}^{-(1+a/\mu)^{1/2}y}\right]\cos x \tag{10.15}$$

$$\tau_{yy} = -A_1\left[\left(\frac{2\mu}{a}+1\right)\mathrm{e}^{-y}-\frac{2\mu}{a}\mathrm{e}^{-(1+a/\mu)^{1/2}y}\right]\cos x \tag{10.16}$$

$$\tau_{xy} = \frac{\mu A_1}{a}\left[-2\mathrm{e}^{-y}+\left(\left(1+\frac{a}{\mu}\right)^{\frac{1}{2}}+\left(1+\frac{a}{\mu}\right)^{-\frac{1}{2}}\right)\mathrm{e}^{-(1+a/\mu)^{1/2}y}\right]\sin x. \tag{10.17}$$

Since the boundary conditions of the solid part are such that the surface tractions are all known, the problem can be solved by use of a stress function. Obtaining $\psi$ from (10.8).

$$\psi = \frac{A_1}{a}\left(1+\frac{a}{\mu}\right)^{-\frac{1}{2}}\mathrm{e}^{-(1+a/\mu)^{1/2}y}\sin x \tag{10.18}$$

equation (6.17) becomes

$$\nabla^4\theta = A_1 h\mathrm{e}^{-(1+a/\mu)^{1/2}y}\cos x \tag{10.19}$$

where

$$h = \left(1+\frac{a}{\mu}\right)-\left(1+\frac{a}{\mu}\right)^{-1}.$$

The solution of the above equation is

$$\theta = \theta_h + A_1 h\frac{\mu^2}{a^2}\mathrm{e}^{-(1+a/\mu)^{1/2}}y\cos x \tag{10.20}$$

where $\theta_h$ is a biharmonic scalar function of $x$ and $y$ and the second term is the particular solution of (10.19). The above problem is equivalent to the similar strip problem of classical elasticity under identical loading except for the extra term coming from the interaction in the form of a body force. The methods of classical elasticity are applicable if the effect of interaction is treated as a prescribed body force. The details of this solution will not be presented here.

The problems of Sections 9 and 10 illustrate the utilization of the solution methods presented in earlier sections. The boundary conditions given in the example in Section 10 are typical of an actual physical problem. It can be easily seen that these boundary conditions, necessary to define a unique physical problem, cannot be satisfied by conventional approaches to the problem of flow through porous media due to the nature of differential equation (Darcy's law) used. In this case, the constants $B_\alpha$ and $C_\alpha$ in (10.8) provide the additional constants required. We noted that the errors involved in neglecting solution (10.8) would diminish as $\alpha/\mu$ becomes large as can be easily seen from the equations. However, the errors remain large for small enough values of $y$ even if $\alpha/\mu$ is large. This shows that the conventional approach, while it might produce good results far enough from the boundaries, would generally fail near the boundaries of the regions under consideration.

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**Абстракт**—Определяются конститутивные уравнения для Взаимодействующей сплошной среды, состоящей из упругого твердого тела и несжимаемой ньютоновой жидкости. Условие несжимаемости изменяет уравнения, полученные раньше Грином и Стилем. Приводится новая концепция термодинамического давления. Новые уравнения сравниваются с такими же, выведенными предыдущими исследователями.

Используя эти уравнения, предлагается метод решения в виде функции перемещений или напряжений для условия стационарного состояния. Настоящие уравнения указывают, что они сводятся к закон Дарси, для течения жидкостей сквозь жесткую пористую среду и к уравнеиню Био, для течения жидкости сквозь линейное, упругое тело. В качестве примеров решаются задачи для одномерного и двухмерного стационарного состояния.